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A stochastic treatment of the dynamics of an integer spin

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Abstract. We prove that, for the general dynamics of a quantum spin, there exists a stochastic process in an extended phase space which at each time allows us to compute correlations between the different components of the spin. The schema is limited to integer spins, but some possibilities of extension are discussed as well as the connection with Nelson's stochastic dynamics.

1. Introduction

Nelson's programme (1967, 1985) for giving a new physical interpretation of the non-relativistic quantum dynamics is fascinating. In particular, its interpretation of the Schrödinger equation in configuration space in terms of the Fokker-Planck equation of a Newtonian diffusion process is quite interesting, both on conceptual grounds and for the possibility of suggesting questions and even tests (see, e.g., Truman and Lewis (1986) and Blanchard *et al* (1987) for a review).

However, Nelson insists on the universality of Brownian motion whose diffusion coefficient is $\sqrt{\hbar}$ in the configuration space description of quantum dynamics (mass is taken equal to one). This causes a problem if one tries to extend the programme to an equivalent description of quantum dynamics, in momentum space for instance (De Angelis 1986) or even to more general polarisations of phase space. However, these questions are natural if one wants to mimic what is done in classical mechanics.

In fact, everything works if one still uses infinitely divisible processes but not diffusions any more. Then it is very difficult to compare processes corresponding to different polarisations.

Another question one can ask oneself is about mixed states (e.g. non-zero temperature states). Almost nothing is known about this problem (see, however, Jaekel and Pignon 1984).

The second question is completely solved by our approach which uses the Wigner function and not the wavefunction. This is in our opinion, the great advantage of this approach since it does not give ambiguities as in the paper quoted above when at least two eigenvalues of the Hamiltonian are degenerate. On the other hand, it treats both the intrinsic and external stochasticity on the same footing.

The first question is not solved except for the case of the dynamics of an integer spin. This case, of course, is not the most interesting one but it has a flavour of the

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continuous case and we hope to extend our results to the case of a continuous system with appropriate cutoffs. As a side remark it is amusing to see that, for our techniques, the requirement that the spin is integer has some flavour of the spin statistics theorem.

Our paper is organised as follows. In § 2 we describe in a convenient way the algebra of observables of a spin s as well as its states. Then we define what we call a dynamics. In § 3 we give the main result (proposition 3.2) for the existence of a stochastic process in an extended phase space which allows us to compute any correlation between different components of the spin. Finally proposition 3.3 gives the characterisation of these processes which are associated with a quantum evolution.

For a general study of Markov processes we refer to the classical books of Gihman and Skorohod (1969).

2. Integer spin system. Observables and states

The system we have in mind is a particle with integer spin s, ignoring all the other degrees of freedom: position, momentum, etc. Of course, this system is quite simple and well known but to formulate the main results we shall prove in the last section that we need an unusual formalism.

First let us describe it in a conventional way. The observable of this system is a spin $\boldsymbol{S}$ with three components $\boldsymbol{S}=\{S_x, S_y, S_z\}$ in an orthonormal basis. These components do not commute but satisfy

$$[S_x, S_y]=\mathrm{i}S_z \tag{2.1}$$

and the cyclic permutations ($\hbar$, the Planck constant, is taken equal to 1).

Furthermore

$$S_x^2+S_y^2+S_z^2=s(s+1)\mathbb{1} \tag{2.2}$$

and

$$\text{spectrum}\{S_z\}=\{-s, -s+1, \ldots, s-1, s\}. \tag{2.3}$$

All this makes the algebra generated by the S_x, S_y and S_z isomorphic to M_{2s+1} (the complex matrices with $2s+1$ columns and lines) acting on the (pure) state space $\mathbb{C}^{2s+1}$.

Due to commutation relations (2.1) only one component, traditionally S_z, can be observed.

Let us introduce an exotic description of the system.

$\mathbb{C}^{2s+1}$ is isomorphic to $\ell_2(\mathbb{Z}_{2s+1})$ where $\mathbb{Z}_{2s+1}$ is the cyclic group of order $2s+1$. Then let us define the following operators for $\varphi\in\ell_2(\mathbb{Z}_{2s+1})$:

$$(W_{mn}\varphi)(k)=\exp\left(-4\frac{\mathrm{i}\pi mn}{2s+1}+4\frac{\mathrm{i}\pi nk}{2s+1}\right)\varphi(k-2m) \tag{2.4}$$

$m, n, k\in\{-s, -s+1, \ldots, s\}$, the operation $k-2m$ being of course interpreted mod $2s+1$.

Lemma 2.1. The operators W_{mn} are unitary and satisfy

$$W_{mn}^*=W_{-m-n} \tag{2.5}$$

$$W_{mn}W_{m'n'}=\exp\left(\frac{4\mathrm{i}\pi}{2s+1}(m'n-mn')\right)W_{m+m'\ n+n'}. \tag{2.6}$$

The proof is obvious.

These operators have the properties of a Weyl system (Weyl 1931) in the continuous case. We call them *Weyl operators.*

The next proposition also gives a property of the Weyl system which is shared by the continuous Weyl operators.

Proposition 2.2. The set of Weyl operators $\{W_{mn}\}_{m,n\in\{-s,\dots,s\}}$ is a system of unitary operators, linearly independent, which generates the algebra $\mathcal{B}(\ell_2(\mathbb{Z}_{2s+1}))$ of operators on $\ell_2(\mathbb{Z}_{2s+1})$.

The proof is almost obvious. It comes essentially from the Stone-von Neumann-Mackey theorem (von Neumann 1955, Mackey 1963) if one notices that the set $\{W_{mn}\}_{m,n\in\{-s,\dots,s\}}$ is a central extension of the group $\mathbb{Z}_{2s+1}\times\mathbb{Z}_{2s+1}$ by the 2-cocycle

$$(m,n)\times(m',n')\rightarrow\exp\left(\frac{4\mathrm{i}\pi}{2s+1}(m'n-mn')\right)$$

and that $2s+1$ is odd. On the other hand, the result can be proved directly if one remarks that the $\{W_{mn}\}$ are orthogonal with respect to the Hilbert-Schmidt product and that their number matches the dimension of $\mathcal{B}(\ell_2(\mathbb{Z}_{2s+1}))$.

$\ell_2(\mathbb{Z}_{2s+1})$ contains different classical orthonormal bases:

$$\{\delta_h\}_{h\in\{-s,\dots,s\}}\qquad\text{such that}\qquad \delta_h(k)=(2s+1)^{1/2}\delta_{h,k}\qquad k\in\{-s,\dots,s\}\tag{2.7}$$

$$\{\chi_h\}_{h\in\{-s,\dots,s\}}\qquad\text{such that}\qquad \chi_h(k)=\exp\left(\frac{2\mathrm{i}\pi kh}{2s+1}\right)\qquad k\in\{-s,\dots,s\}.\tag{2.8}$$

These bases are exchanged by the Fourier transform $\mathcal{F}$ from $\ell_2(\mathbb{Z}_{2s+1})$ to $\ell_2(\mathbb{Z}_{2s+1})$:

$$(\mathcal{F}\varphi)(k)=\frac{1}{(2s+1)^{1/2}}\sum_{h=-s}^{s}\exp\left(-\frac{2\mathrm{i}\pi}{2s+1}kh\right)\varphi(h).\tag{2.9}$$

$\mathcal{F}$ is an isometry and satisfies with respect to the Weyl operators

$$\mathcal{F}W_{m,n}\mathcal{F}^*=W_{n,-m}\qquad m,n\in\{-s,\dots,s\}.\tag{2.10}$$

A state of the system is given by a density matrix ρ on $\ell_2(\mathbb{Z}_{2s+1})$ such that

$$\rho\geqslant 0\tag{2.11}$$

$$\mathrm{Tr}(\rho)=1.\tag{2.12}$$

Equivalently it is completely defined by the function $\Xi(m,n)$ from $(\mathbb{Z}_{2s+1})^2$ to $\mathbb{C}$

$$\Xi(m,n)=\frac{1}{2s+1}\mathrm{Tr}(\rho W_{mn}).\tag{2.13}$$

Indeed ρ can be decomposed on the W_{mn} which are independent but also orthogonal with respect to the trace, hence the following proposition.

Proposition 2.3. If Ξ is a function from $(\mathbb{Z}_{2s+1})^2$ to $\mathbb{C}$ the necessary and sufficient condition for Ξ to be of the form (2.13) is

$$\Xi(0,0)=\frac{1}{2s+1}\tag{2.14}$$

$$\sum_{i,j=1}^{N}\lambda_i\bar{\lambda}_j\exp\left(-\frac{4\mathrm{i}\pi}{2s+1}(m_in_j-m_jn_i)\right)\Xi(m_i-m_j,n_i-n_j)\geqslant 0\tag{2.15}$$

for all N and for all choices of $\{\lambda_i\}_{i=1\dots N}$, $\lambda_i\in\mathbb{C}$. Again, the proof is obvious.

A pure state, namely the case where ρ is a projection on the normalised vector $\varphi \in \ell_2(\mathbb{Z}_{2s+1})$, corresponds to the following Ξ^{φ}:

$$\Xi^{\varphi}(m,n) = \exp\left(-\frac{4\mathrm{i}\pi}{2s+1}mn\right)\frac{1}{2s+1}\sum_{h=-s}^{s}\bar{\varphi}(h)\exp\left(\frac{4\mathrm{i}\pi nh}{2s+1}\right)\varphi(h-2m). \tag{2.16}$$

As in the continuous case one has the following proposition (Fano 1957, Grossmann 1976, Combe *et al* 1983).

Proposition 2.4. Let P be the symmetry in $\ell_2(\mathbb{Z}_{2s+1})$:

$$(P\varphi)(k) = \varphi(-k) \qquad k \in \{-s, \ldots, s\}$$

and let ρ be a density matrix on $\ell_2(\mathbb{Z}_{2s+1})$. Then

$$\frac{1}{2s+1}\sum_{m=-s}^{s}\sum_{n=-s}^{s}\exp\left(\frac{4\mathrm{i}\pi}{2s+1}(mb-na)\right)\mathrm{Tr}(\rho W_{mn})$$
$$= \mathrm{Tr}\{\rho W_{ab}P\} \qquad a, b \in \{-s, \ldots, s\}. \tag{2.17}$$

The proof is a matter of computation.

Then one can define the analogue of the Wigner functions (Wigner 1932).

Definition 2.5. If ρ is a density matrix then

$$\mathscr{W}(a,b) = \frac{1}{2s+1}\mathrm{Tr}(\rho W_{ab}P) \qquad a, b \in \{-s, \ldots, s\} \tag{2.18}$$

as a function of $(\mathbb{Z}_{2s+1})^2$ is a *Wigner function.*

In the case of a pure state where ρ corresponds to the orthogonal projection on the normalised vector φ of $\ell_2(\mathbb{Z}_{2s+1})$ it has the expression

$$\mathscr{W}(a,b) = \frac{1}{2s+1}\sum_{h=-s}^{s}\exp\left(\frac{4\mathrm{i}\pi}{2s+1}hb\right)\bar{\varphi}(h+a)\varphi(a-h). \tag{2.19}$$

As in the continuous case this suggests we define new operators, the *Fano operators*, as below.

Definition 2.6 (*Fano 1957, Grossmann 1976*). The Fano operators Δ_{ab} are defined by

$$\Delta_{ab} = W_{ab}P \qquad a, b \in \{-s, \ldots, s\}. \tag{2.20}$$

Proposition 2.7. The Fano operators satisfy the following properties:

$$\Delta^*_{ab} = \Delta_{ab} \tag{2.21}$$

$$(\Delta_{ab})^2 = \mathbb{1}_{2s+1} \tag{2.22}$$

$$\mathrm{Tr}(\Delta^*_{ab}\Delta_{a'b'}) = (2s+1)\delta_{a,a'}\delta_{b,b'} \tag{2.23}$$

$$\Delta_{ab}\Delta_{a'b'} = \exp\left(\frac{8\mathrm{i}\pi}{2s+1}(ab'-ba')\right)\Delta_{-a'-b'}\Delta_{-a-b} \tag{2.24}$$

$$W^*_{ab}\Delta_{mn}W_{ab} = \Delta_{m-2a\ n-2b}. \tag{2.25}$$

All these properties are obvious.

Finally, Wigner functions have properties which are the same as in the continuous case.

Proposition 2.8. If $\mathcal{W}$ is a Wigner function then

(i) $\mathcal{W}$ is real valued,

(ii) $\mathcal{W}$ is bounded by $1/(2s+1)$ in modulus,

(iii) $\Sigma_{m,n\in\{-s,\dots,s\}} \mathcal{W}\{m, n\} = 1$,

(iv) $\mathcal{W}$ is the symplectic Fourier transform of a ξ-positive function (defined by form (2.14) and (2.15)).

Nevertheless $\mathcal{W}$ is not a probability density on $(\mathbb{Z}_{2s+1})^2$ since there are explicit cases (easy to construct) where it is negative.

Before closing the section let us quote two results which are important for the following.

Lemma 2.9. If $\mathcal{W}$ is a Wigner function such that

$$\mathcal{W}(a, b) = \frac{1}{2s+1} \operatorname{Tr}(\rho\Delta_{ab}) \tag{2.26}$$

then ρ can be reconstructed as

$$\rho = \sum_{a,b\in\{-s,\dots,s\}} \mathcal{W}(a, b)\Delta_{ab} \tag{2.27}$$

and moreover formula (2.27) defines a density matrix associated with any function $\mathcal{W}$ which is the symplectic Fourier transform of a function satisfying (2.14) and (2.15). Hence correspondence between density matrices and Wigner functions is bijective.

Lemma 2.10. If $\mathcal{W}$ is a Wigner function whose density matrix is the orthogonal projection on the normalised vector φ of $\ell_2(\mathbb{Z}_{2s+1})$ then

$$\sum_{b=-s}^{s} \mathcal{W}(a, b) = |\varphi(a)|^2 \tag{2.28}$$

$$\sum_{a=-s}^{s} \mathcal{W}(a, b) = |\mathcal{F}\varphi(b)|^2. \tag{2.29}$$

Again the proof is a matter of computation.

Now we shall describe in this language the dynamics. It is given by a Hamiltonian operator $H \in \mathcal{B}(\ell_2(\mathbb{Z}_{2s+1}))$ which defines the evolution equation of any density matrix ρ_t, $t \in R$, through the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\mathrm{i}[H, \rho_t] \tag{2.30}$$

plus an initial condition or, equivalently, of any 'wavefunction' φ in $\ell_2(\mathbb{Z}_{2s+1})$ through the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = H\varphi_t \tag{2.31}$$

plus an initial condition.

H belonging to $\mathcal{B}(\ell_2(\mathbb{Z}_{2s+1}))$ has a decomposition both in terms of the Weyl operators and in term of the Fano operators:

$$H = \sum_{q,p\in\{-s,\dots,s\}} \mathcal{H}(q, p)\Delta_{qp} \tag{2.32}$$

$$H = \sum_{a,b\in\{-s,\dots,s\}} \tilde{\mathcal{H}}(a, b)\, W_{ab} \tag{2.33}$$

where $\tilde{\mathscr{H}}(a, b) = \hat{\mathscr{H}}(2a, 2b)$, $\hat{\mathscr{H}}$ being the symplectic Fourier transform of $\mathscr{H}$:

$$\hat{A}(a, b) = \frac{1}{2s+1} \sum_{q,p \in \{-s,\dots,s\}} \exp\left(-\frac{2\mathrm{i}\pi}{2s+1}(bq - ap)\right) A(q, p).$$

$\mathscr{H}$ is real in order to define a conservative dynamics. For the sake of simplicity let us assume that $\tilde{\mathscr{H}}$ is positive. One can do without this assumption. However, the formulae become cumbersome to deal with.

The evolution equation (2.30) has an immediate transcription for the time evolution of Wigner functions, viz

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{W}_t(a, b) = \sum_{a',b' \in \{-s,\dots,s\}} \mathrm{i}\tilde{\mathscr{H}}(a', b') \exp\left(\frac{4\mathrm{i}\pi}{2s+1}(b'a - a'b)\right) \times \{\mathscr{W}_t(a + a', b + b') - \mathscr{W}_t(a - a', b - b')\} \tag{2.34}$$

plus an initial condition.

In his original paper Wigner (1932) stresses that this equation looks like the forward Kolmogorov equation of a jump process in phase space. However two things prevent us from going further in this direction:

(i) transition probabilities would be negative, and

(ii) $\mathscr{W}$ is not a probability density since it can assume negative values.

Another possibility would be of interpreting (2.35) as the backward Kolmogorov equation of a stochastic process. This cannot be done in the phase space $(\mathbb{Z}_{2s+1})^2$ itself but in an extended one by $S_2 = \{\pm 1\}$. This schema has been successfully developed in Combe *et al* (1984) and Blanchard and Sirugue (1985).

This approach, however, is not in the spirit of Nelson's stochastic mechanics and in the next section we show that a stochastic scheme interpreting (2.34) as the trace of a forward Kolmogorov equation for a jump process in an extended phase space does exist.

3. Stochastic mechanics in extended phase space

The purpose of this section is to prove the main result of this paper, namely, given a quantum evolution of the spin, there exists in an extended phase space a stochastic process which allows us to compute at each time correlations between the different components of the spin. It does not reproduce the correlations themselves since quantum theory cannot be reduced to a classical theory (von Neumann 1955).

Let us be more precise. Let σ is a dichotomic variable, viz $\sigma \in \{\pm 1\}$. Then, if $\mathscr{W}$ is a Wigner function

$$G(m, n, \sigma) = \frac{1}{4(2s+1)}\left(\frac{2}{2s+1} + \sigma \mathscr{W}(m, n)\right) \tag{3.1}$$

is a function of $(\mathbb{Z}_{2s+1})^2 \times \{\pm 1\}$ with the following properties.

(i) $G(m, n, \sigma)$ is real, positive, and furthermore (3.2)

$$\frac{1}{4(2s+1)^2} \leqslant G(m, n, \sigma) \leqslant \frac{3}{4(2s+1)^2}$$

(ii)
$$\sum_{\substack{m,n \in \{-s,\dots,s\} \\ \sigma = \pm 1}} G(m, n, \sigma) = 1 \tag{3.3}$$

(iii) if f is a function on $(\mathbb{Z}_{2s+1})^2$ then the quantum expectation value of f is just

$$\langle f\rangle_Q = 2(2s+1) \sum_{\substack{m,n\in\{-s,\ldots,s\}\\ \sigma=\pm1}} \sigma G(m,n,\sigma) f(m,n). \tag{3.4}$$

These properties show that G can be interpreted as a probability density on extended phase space $(\mathbb{Z}_{2s+1})^2 \times \{\pm 1\}$ and that it allows us to compute any quantum correlation $\mathcal{W}(m,n)$ between different components of the spin, at a given time.

The next remark is almost obvious.

Lemma 3.1. If $t \to \mathcal{W}_t(m,n)$ is a family of Wigner functions which satisfy an evolution equation (2.34) the associated G_t function obeys the same equation which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} G_t(m,n,\sigma) = \sum_{m',n'\in\{-s,\ldots,s\}} -2\sin\left(\frac{4\pi}{2s+1}(mn'-m'n)\right)\tilde{\mathcal{H}}(m'-m, n'-n) G_t(m',n',\sigma). \tag{3.5}$$

Once this is done it is sufficient to compute the generator of the process to use the technique of Guerra and Marra (1984) to transform (3.5) into a forward Kolmogorov equation for a stochastic process which is, of course, state dependent.

Proposition 3.2. There exists a stochastic Markov process X_t living in $(\mathbb{Z}_{2s+1})^2 \times \{\pm 1\}$ whose density is given at each time by $G_t(m,n,\sigma)$ and whose forward Kolmogorov equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} G_t(m,n,\sigma) = \sum_{\substack{m',n'\in\{-s,\ldots,s\}\\ \sigma'\in\{\pm1\}}} A_t(m,n,\sigma; m',n',\sigma') G_t(m',n',\sigma') \tag{3.6}$$

where

$$\begin{aligned} A_t(m,n,\sigma;m',n',\sigma') &= 2\left[\frac{1}{G_t(m',n',\sigma')} - \delta_{\sigma,\sigma'}\sin\left(\frac{4\pi}{2s+1}(mn'-m'n)\right)\right] \\ &\quad \times \tilde{\mathcal{H}}(m'-m, n'-n) \qquad \text{if } (m',n') \neq (m,n) \\ A_t(m,n,\sigma;m,n,\sigma') &= 0 \qquad \text{if } \sigma \neq \sigma' \\ A_t(m,n,\sigma;m,n,\sigma) &= -\frac{4}{G_t(m,n,\sigma)} \sum_{(m',n')\in(\mathbb{Z}_{2s+1})^2\setminus(0,0)} \tilde{\mathcal{H}}(m',n'). \end{aligned} \tag{3.7}$$

The proof is a matter of computation.

To be complete we want to characterise those Markov processes which are associated with a quantum evolution. This is done in the next proposition.

Proposition 3.3. Let $A_t(m,n,\sigma;m',n',\sigma')$ be the generator of a Markov process in $(\mathbb{Z}_{2s+1})^2 \times \{\pm 1\}$ and $G_t(m,n,\sigma)$ be the probability density of this process. Then the necessary and sufficient condition for the existence of a quantum dynamics in the previous sense is the existence of an operator B on $\mathcal{B}(\ell_2((\mathbb{Z}_{2s+1})^2))$ such that

$$\begin{aligned} &\sum_{\sigma,\sigma'\in\{\pm1\}} \sigma A_t(m,n,\sigma;m',n',\sigma') G_t(m',n',\sigma') \\ &\qquad = \sum_{\sigma'\in\{\pm1\}} \sigma' B_{m,n;m',n'} G_t(m',n',\sigma') \end{aligned} \tag{3.8}$$

and an operator H of $\mathscr{B}(\ell_2(\mathbb{Z}_{2s+1}))$ such that

$$\sum_{(m',n')\in(\mathbb{Z}_{2s+1})^2} B_{m,n;m',n'}\Delta_{m'n'} = \mathrm{i}[H, \Delta_{mn}]. \tag{3.9}$$

The proof is tedious but can be done if one remarks that H allows us to reconstruct the Hamiltonian flow in $\ell_2(\mathbb{Z}_{2s+1})$ by the Dyson series.

To be complete let us remark that, if it exists, $\tilde{\mathscr{H}}$ is given by

$$\tilde{\mathscr{H}}(m, n) = -\frac{\mathrm{i}}{(2s+1)^2} \sum_{m',n'\in\mathbb{Z}_{2s+1}} \exp\left(\frac{4\mathrm{i}\pi}{2s+1}(m'n - mn')\right) B_{m,n;m+m',n+n'}. \tag{3.10}$$

In this scheme the quantum mechanics of an integer spin appears as the mixture of two classical schemes of a spin. However at random times the schemes are exchanged and that prevents us reducing the quantum theory to a classical one.

The extension of the previous results to half-integer spin is not directly possible since the natural group $\mathbb{Z}_{2s}$ is not a divisible group. That would amount to dealing with fermions rather than bosons.

The continuous case cannot be treated, at least directly, with such a technique. Indeed, phase space is no longer compact and the corresponding G_t one can define out of a Wigner function is no longer a probability density. However, one can deal with non-bounded positive measures but then, if everything works formally, from a technical point of view it is difficult in a rigorous way to prove the existence of the stochastic Markov processes (Cohendet 1987).

As a last remark the choice of $\{\pm 1\}$ for the extra variable σ is not the only choice one can imagine provided it is a compact space, e.g. the torus would work as well. However, in view of the measure one has to put on this space, $\{\pm 1\}$ is a minimal choice.

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